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Homoclinic Bifurcation Sets of Driven Nonlinear Oscillators

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Different quasiperiodically and parametrically driven nonlinear oscillators with quadratic and cubic nonlinearities are considered, and the corresponding homoclinic bifurcation sets in a five-dimensional parameter space are explicitly calculated. We classify all these cases into two basic types of homoclinic bifurcation sets: the first one corresponds to quasiperiodically driven oscillators and the second one corresponds to parametrically driven oscillators.

Periodically driven nonlinear oscillators have attracted much study in the past few years (Moon, 1992). One of the main reasons for this attention is the rich dynamic behavior observed in them and the enormous applications of these nonlinear oscillators for modeling oscillatory and complex phenomena in all branches of science. Besides periodically driven nonlinear oscillators, much attention also has been paid recently to quasiperiodically and parametrically driven nonlinear oscillators (Ide and Wiggins, 1989; Parthasarathy, 1992; McLaughlin, 1981; Koch and Leven, 1985; Wiggins, 1987; Lima and Pettini, 1990; Yagasaki, 1992, 1994; Kapitaniak, 1993; Cicogna and Fronzoni, 1993; Cuadros and Chacón, 1993; Kivshar *et al.*, 1994).

The Mełnikov theory is the analytical tool that has been used most in order to ascertain the critical parameter values for which a system is expected to show chaotic behavior of the Smale-horseshoe type (Wiggins, 1990). In spite of the power of the method in predicting the chaotic threshold, it is important to note that in practice the true observed threshold is above the predicted one and this is mainly due to its intrinsic perturbative nature (Kivshar *et al.*, 1994; Lima and Pettini, 1993; Grauer *et al.*, 1993).

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Ide and Wiggins (1989) (referred to as IW) study the homoclinic bifurcation sets of the quasiperiodically forced Duffing oscillator, and Parthasarathy (1992) (referred to as P) does the same for the case of the parametrically driven Duffing oscillator. Both of them use basically the same method, i.e., they begin by writing the original equation as a set of two coupled first-order differential equations in suspended form, then apply the Melnikov technique, evaluating the Melnikov function which depends upon the different parameters of the original nonlinear oscillator. Finally, a criterion for the occurrence of chaos of the Smale-horseshoe type is established for the damping coefficient, say k, in such a way that whenever this coefficient is less than a critical value k_c, then transverse intersections of the stable and the unstable manifolds occur, and the attendant chaotic dynamics is expected. Both studies construct explicitly the corresponding homoclinic bifurcation sets in the five-dimensional parameter space, say $(k, f_1, f_2, \omega_1, \omega_2)$, where k is the damping coefficient, f_1 and f_2 are the respective parameters of the forcing, and ω_1 and ω_2 are the corresponding associated frequencies. This homoclinic bifurcation set is given by the four-dimensional surface which is obtained through the mathematical condition provided by the Melnikov method. In case IW, f_1 and f_2 represent the two competing quasiperiodic external forces, while in case P, f_1 and f_2 represent competition between the parametric forcing and the external forcing.

To represent these bifurcation curves in the ω_1–ω_2 plane for a chosen set of values of (k, f_1, f_2) we define two functions, say $X_1(\omega_1)$ and $X_2(\omega_2)$, in such a way that the mathematical condition for the threshold of chaotic behavior can be rewritten basically as

$$-k + f_1 X_1(\omega_1) + f_2 X_2(\omega_2) = 0 \qquad (1)$$

Since this equation is linear in f_1, f_2, $X_1(\omega_1)$, and $X_2(\omega_2)$, it can be regarded as a surface in the five-dimensional $(k, f_1, f_2, \omega_1, \omega_2)$ space. The bifurcation curves in the X_1–X_2 plane are just lines. Then, using the properties of the functions $X_1(\omega_1)$ and $X_2(\omega_2)$ and their inverses, it is possible to redraw the bifurcation curves in the ω_1–ω_2 plane from the lines in the X_1–X_2 plane. Finally the complete homoclinic bifurcation sets of the corresponding oscillator are drawn in the X_1–X_2 plane and in the ω_1–ω_2 plane for the different cases in the (k, f_1, f_2) space. More details concerning this construction can be found in Ide and Wiggins (1989) and Parthasarathy (1992). The main difference between the two analyses lies in the different nature of the functions $X_1(\omega_1)$ and $X_2(\omega_2)$. While in case P both functions possess different maxima at, say, ω_1' and ω_2', in case IW both functions possess their maxima at the same frequency ω_m. This results in some different bifurcations sets in the ω_1–ω_2 plane, along with some additional subcases for case P, as is shown in Parthasarathy (1992).

In the present paper we analyze and explicitly calculate the homoclinic bifurcation sets for different quasiperiodically and parametrically driven nonlinear oscillators with quadratic and cubic nonlinearities. We start with the Helmholtz oscillator (Rasband, 1987; Thompson, 1989), which is a nonlinear oscillator with a quadratic nonlinearity and which appears in different fields of science. It possesses a quadratic nonlinearity and also appears in connection with steady reductions of some perturbative KdV equations governing nonlinear waves and solitons (Grimshaw and Tian, 1994; Sanjuán, n.d.; Zimmermann and Velarde, n.d.). The equation of motion for the parametrically driven Helmholtz oscillator with a periodic external forcing and a parametric modulation acting in the linear dynamical variable is given by

$$\ddot{x} + k\dot{x} - \alpha(1 + \gamma \sin \omega_2 t)x + \beta x^2 = F \sin \omega_1 t \tag{2}$$

where α, β, γ, k, F, ω_2, and ω_1 are positive constants and $0 < \gamma \leq 1$. For the unperturbed system, i.e., when $\gamma = k = F = 0$, we obtain the conservative Helmholtz oscillator, whose Hamiltonian can be written as

$$H(\dot{x}, x) = \frac{\dot{x}^2}{2} - \frac{\alpha x^2}{2} + \frac{\beta x^3}{3} \tag{3}$$

This system has a center in $(\alpha/\beta, 0)$ and a saddle in $(0, 0)$ in phase space. The equations for the homoclinic orbit are

$$x_{sx}(t) = \frac{3\alpha}{2\beta} - \frac{3\alpha}{2\beta} \tanh^2\left\{\left(\frac{\alpha}{4}\right)^{1/2} t\right\} \tag{4}$$

$$y_{sx}(t) = -\frac{3}{2}\left(\frac{\alpha^3}{\beta^2}\right)^{1/2} \frac{\sinh\{(\alpha/4)^{1/2}t\}}{\cosh^3\{(\alpha/4)^{1/2}t\}} \tag{5}$$

This shows that there are different possible motions. There are bounded motions in the interior of the separatrix and there are unbounded motions outside the separatrix. The presence of the perturbations added to the oscillator causes the stable and unstable manifolds to be destroyed, giving rise to the possibility of chaotic solutions. We are interested in the calculation of the Melnikov distance $\Delta(t_0)$ for the case in which all the perturbations are considered. A transformation of k, γ, and F into ϵk, $\epsilon\gamma$, and ϵF is done in order to apply the first-order perturbation scheme of the Melnikov theory. Equation (2) may be written as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \alpha x - \beta x^2 + \epsilon\{F \sin \omega_1 t + \alpha\gamma x_{sx}(t) \sin \omega_2 t - k y_{sx}(t)\} \end{aligned} \tag{6}$$

The Melnikov distance is evaluated as

$$\Delta(t_0) = \int_{-\infty}^{+\infty} y_{sx}(t)\{F \sin \omega_1(t + t_0) + \alpha\gamma x_{sx}(t)\sin \omega_2(t + t_0) - k y_{sx}(t)\}\, dt \tag{7}$$

which can be written in two parts, $\Delta_0(t_0)$ and $\Delta_\gamma(t_0)$. The first term corresponds to the case in which there is only external forcing, while the second term is the contribution of the parametric forcing. The computation of the Melnikov distance then gives

$$\begin{aligned}\Delta(t_0) &= \Delta_0(t_0) + \Delta_\gamma(t_0)\\ &= \frac{6\pi}{\beta}\frac{\omega^2 F \cos \omega_1 t_0}{\sinh[\pi(4/\alpha)^{1/2}\omega_1/2]} - \frac{6\alpha^{5/2}}{5\beta^2}k + \frac{3\pi\alpha^2}{2\beta^2}\frac{\omega_2^2\gamma \cos \omega_2 t_0}{\sinh[\pi(4/\alpha)^{1/2}\omega_2/2]}\\ &\quad \left\{4 + \frac{\omega_2^2}{\alpha}\right\}\end{aligned} \tag{8}$$

The condition for transverse intersection and chaotic separatrix motion holds when $\Delta(t_0)$ changes sign at some t_0 (Wiggins, 1990). This criterion for the appearance of chaos can be finally written, for the case in which $\alpha = \beta = 1$, in the form

$$k \leq k_c = \frac{5\pi F\omega_1^2}{\sinh(\pi\omega_1)} + \frac{5\pi\gamma\omega_2^2}{4 \sinh(\pi\omega_2)}\{4 + \omega_2^2\} \tag{9}$$

If we define $f_1 = 5\pi F$, $f_2 = 5\pi\gamma/4$, $X_1(\omega_1) = \omega_1^2/\sinh(\pi\omega_1)$, and $X_2(\omega_2) = \omega_2^2\{4 + \omega_2^2\}/\sinh(\pi\omega_2)$, then this last equation can be rewritten in the form

$$-k + f_1 X_1(\omega_1) + f_2 X_2(\omega_2) = 0 \tag{10}$$

which is formally the same as equation (A.1) of Ide and Wiggins (1989) or equation (7b) of Parthasarathy (1992). It can be easily seen that the functions $X_1(\omega_1)$ and $X_2(\omega_2)$ so defined have different maxima at different values of the frequencies, thus resulting in type P. This can be observed in Fig. 1. Figure 1a represents the plots of $X_1(\omega_1)$ and $X_2(\omega_2)$, showing their respective maxima X_1' and X_2' at different frequencies ω_1' and ω_2', while their inverses can be observed in Fig. 1b. The remaining analysis of the homoclinic bifurcation sets for the parametrically driven Helmholtz oscillator is then qualitatively the same as the one given by Parthasarathy (1992). This is due to the similitude of the problem and also to the qualitative nature of the analysis carried out there.

We consider now another example related to the problem just analyzed. Instead of the parametric drive on the dynamic variable, we add another

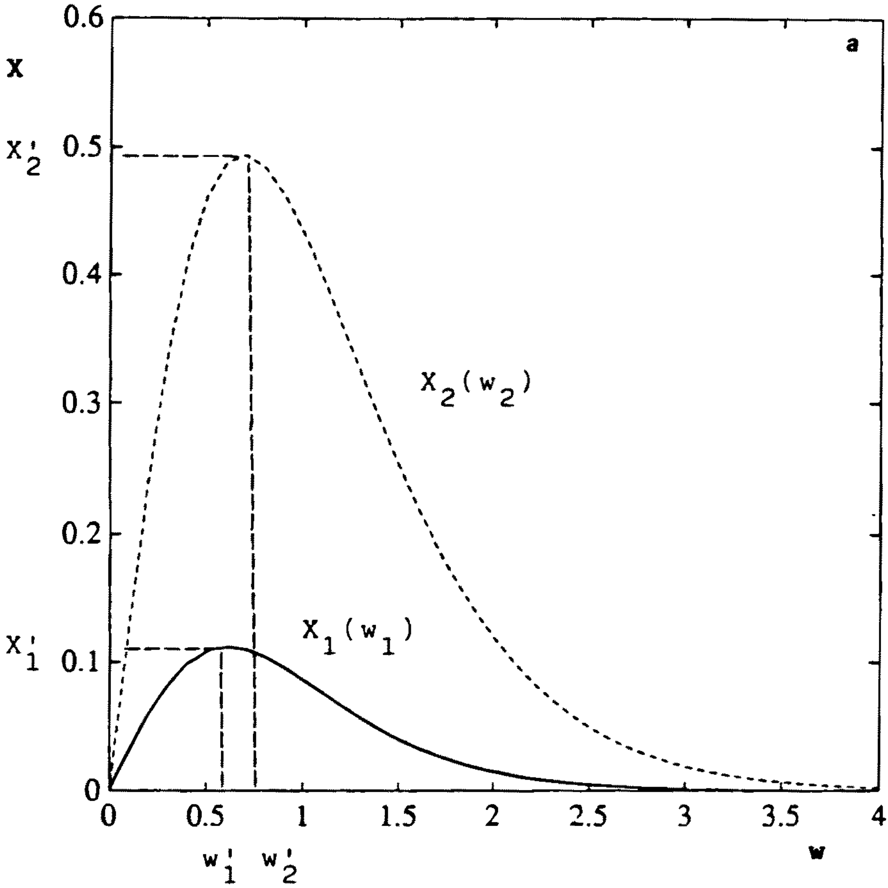

Fig. 1 (a) Plots of the auxiliary functions $X_1(\omega_1)$ and $X_2(\omega_2)$ and (b) their inverses. The most important feature is that the two functions have maxima at different frequencies.

external periodic forcing in such a way that the equation of motion of the quasiperiodically forced Helmholtz oscillator becomes

$$\ddot{x} + k\dot{x} - x + x^2 = f_1 \sin \omega_1 t + f_2 \sin \omega_2 t \tag{11}$$

The expression for the bifurcation set obtained through the Melnikov analysis is given by

$$-k + 5\pi f_1 \omega_1^2 \operatorname{csch}(\pi\omega_2) + 5\pi f_2 \omega_2^2 \operatorname{csch}(\pi\omega_2) = 0 \tag{12}$$

which is qualitatively identical to the expression found in Ide and Wiggins (1989), implying that the homoclinic bifurcation curves are identical.

Besides the cases of nonlinear oscillators we have considered so far, we analyze the parametrically driven Duffing oscillator (Cicogna and Fronzoni,

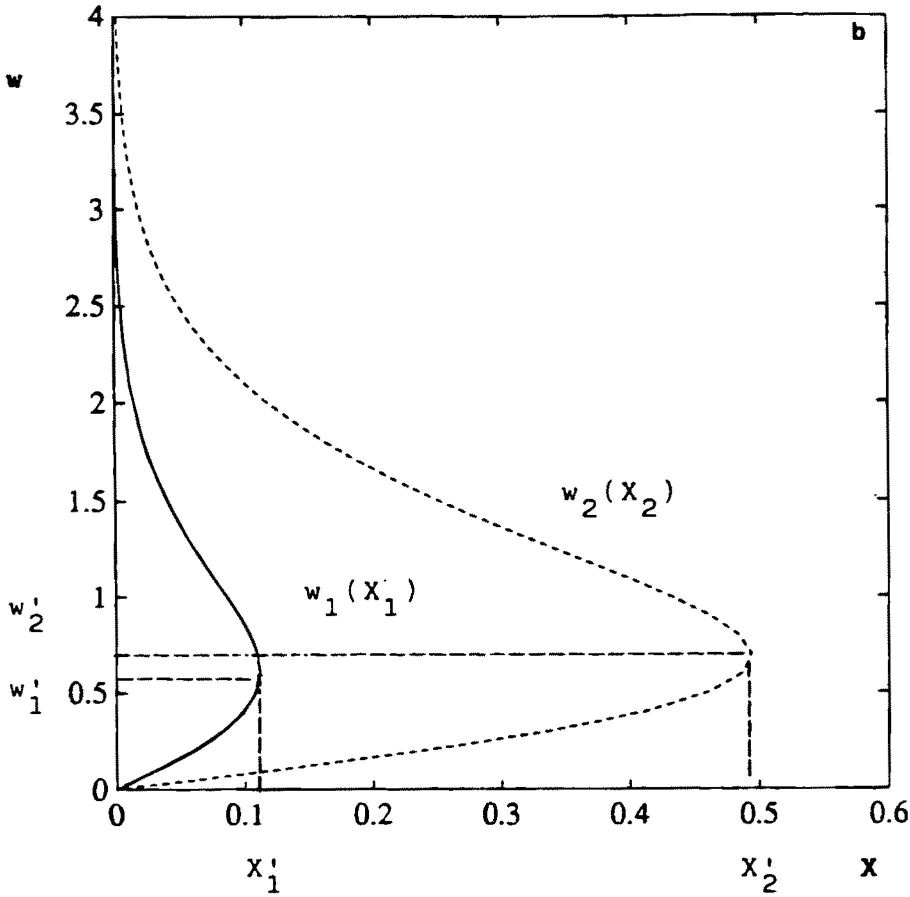

Fig. 1. Continued.

1993; Cuadros and Chacón, 1993), in which both the linear and the nonlinear dynamical variables are modulated. The equation of motion reads

$$\ddot{x} + k\dot{x} - (1 + \epsilon_1 \sin \Omega_1 t)x + (1 + \epsilon_3 \sin \Omega_3 t)x^3 = F \sin \omega t \qquad (13)$$

The computation of the Melnikov function, which provides the equation of the bifurcation set, gives

$$\Delta(t_0) = \sqrt{2}F\pi\omega \operatorname{sech}\left(\frac{\pi\omega}{2}\right) \sin \omega t_0 - \frac{4k}{3} + \pi\epsilon_1\Omega_1^2 \operatorname{csch}\left(\frac{\pi\Omega_1}{2}\right) \sin \Omega_1 t_0 - \frac{\pi\epsilon_3\Omega_3^2(\Omega_3^2 + 4)}{6} \operatorname{csch}\left(\frac{\pi\Omega_3}{2}\right) \sin \Omega_3 t_0 \qquad (14)$$

Different possibilities can exist here, with the constraint of a five-dimensional parameter space. Our possibility is the one considered by Parthasarathy (1992). Another one is when we take into account the external forcing and the parametric forcing acting in the cubic term. The equation of the bifurcation set resulting from this case is of the same kind. If only both parametric forcings are acting, with no external forcing, the result is also of the same kind.

A similar expression is obtained for the case of the parametrically driven Helmholtz oscillator in which both the linear and the nonlinear components are modulated.

After having considered all the different cases of nonlinear oscillators with quadratic and cubic nonlinearities presented here, we can conclude that there exist two basic types of homoclinic bifurcation sets, type IW and type P. If we have a quasiperiodically driven nonlinear oscillator, we obtain a bifurcation set from which a homoclinic bifurcation set of type IW is derived. However, if we consider the parametric drive of one component of the state variable, the linear or the nonlinear, along with the external forcing, or just the parametric drive of both state variables without the external forcing, then the homoclinic bifurcation set for the nonlinear oscillator is of type P. This is nothing strange, because the calculation of the bifurcation set depends directly on the computation of the Melnikov integral. For the quasiperiodically driven systems this integral involves integrands of the same nature, and this is why the result is of type IW. For the parametrically driven systems, the Melnikov integral involves terms of a different nature, since the parametric terms involve the state variable, resulting then in a bifurcation set of type P. As a consequence of this analysis, we conclude that there are two basic types of homoclinic bifurcation sets, one which corresponds to quasiperiodic forcing and the other which corresponds to parametric forcing.

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